

## ON BAIRE-1/4 FUNCTIONS

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ABSTRACT. We give descriptions of the spaces  $D(K)$  (i.e. the space of differences of bounded semicontinuous functions on  $K$ ) and especially of  $B_{1/4}(K)$  (defined by Haydon, Odell and Rosenthal) as well as for the norms which are defined on them. For example, it is proved that a bounded function on a metric space  $K$  belongs to  $B_{1/4}(K)$  if and only if the  $\omega^{\text{th}}$ -oscillation,  $\text{osc}_\omega f$ , of  $f$  is bounded and in this case  $\|f\|_{1/4} = \| |f| + \widetilde{\text{osc}}_\omega f \|_\infty$ . Also, we classify  $B_{1/4}(K)$  into a decreasing family  $(S_\xi(K))_{1 \leq \xi < \omega_1}$  of Banach spaces whose intersection is equal to  $D(K)$  and  $S_1(K) = B_{1/4}(K)$ . These spaces are characterized by spreading models of order  $\xi$  equivalent to the summing basis of  $c_0$ , and for every function  $f$  in  $S_\xi(K)$  it is valid that  $\text{osc}_{\omega^\xi} f$  is bounded. Finally, using the notion of null-coefficient of order  $\xi$  sequence, we characterize the Baire-1 functions not belonging to  $S_\xi(K)$ .

### INTRODUCTION

In recent years the study of the first Baire class,  $B_1(K)$ , of bounded functions on a metric space  $K$  led to the definition of interesting subclasses ([H-O-R], [K-L], [F1]). The study of these subclasses revealed significant properties of their elements ([C-M-R], [R2], [F1], [F2]) and provided applications, such as the  $c_0$ -dichotomy theorem of Rosenthal ([R1]).

Here we study some subclasses of  $D(K)$ , and especially  $B_{1/4}(K)$ , of  $B_1(K)$ . By  $D(K)$  is denoted the class of all functions on  $K$  which are differences of bounded semicontinuous functions. A classical result of Baire yields that  $f \in D(K)$  if and only if there exists a sequence  $(f_n)$  of continuous functions on  $K$  satisfying

$$(1) \quad \sup_{x \in K} \sum_n |f_n(x)| < \infty \quad \text{and} \quad f = \sum_n f_n.$$

The class  $D(K)$  is a Banach algebra with respect to the  $\|\cdot\|_D$ -norm defined as

$$\|f\|_D = \inf \left\{ \sup_{x \in K} \sum_n |f_n(x)| : (f_n) \subseteq C(K) \text{ satisfying (1)} \right\}.$$

The subclass  $B_{1/4}(K)$  was first defined in [H-O-R] as follows:

$$B_{1/4}(K) = \{f : K \rightarrow \mathbf{R} : \text{there exists } (F_n) \subseteq D(K) \text{ such that } \|F_n - f\|_\infty \rightarrow 0 \\ \text{and } \sup_n \|F_n\|_D < \infty\}.$$

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Received by the editors August 22, 1994.

1991 *Mathematics Subject Classification*. Primary 46B03; Secondary 46B25.

This class is a Banach algebra with respect to the  $\|\cdot\|_{1/4}$ -norm, given by

$$\|f\|_{1/4} = \inf \left\{ \sup_n \|F_n\|_D : (F_n) \subseteq D(K) \text{ and } \|F_n - f\|_\infty \rightarrow 0 \right\}.$$

In the first section we describe the precise connection between the summing basis  $(s_n)$  of  $c_0$  and the normed space  $(D(K), \|\cdot\|_D)$ ; so it is proved in Proposition 1.1 that  $f \in D(K)$  if and only if there is a sequence  $(f_n)$  of continuous functions on  $K$  so that  $f_n \rightarrow f$  pointwise and there is  $C > 0$  such that

$$(2) \quad \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty$$

for every  $k, n_1, \dots, n_k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$ .

If this occurs then

$$\|f\|_D = \inf \left\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ satisfying (2)} \right\}.$$

Since for every sequence of continuous functions defined on a compact metric space  $K$  and converging pointwise to a discontinuous function, there exists a subsequence  $(f_n)$  and  $\mu > 0$  such that

$$(3) \quad \mu \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty \leq \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty$$

for  $k, n_1, \dots, n_k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$  ([H-O-R], [R1]), it follows that the functions in  $D(K) \setminus C(K)$  ( $K$  compact) are characterized as pointwise limits of sequences of continuous functions equivalent to the summing basis of  $c_0$  (Remark 1.2).

In the case of  $B_{1/4}(K)$ , where  $K$  is a compact metric space, the functions are characterized as pointwise limits of sequences of continuous functions on  $K$  with a property weaker than (2), namely one for which the inequality (2) is valid only for  $(n_1, \dots, n_k)$  in the Schreier family  $\mathcal{F}_1$  (Theorem 2.1). Moreover, if we set

$$\|f\|_s^1 = \inf \left\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } \begin{array}{l} f_n \rightarrow f \\ \text{pointwise and } \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty \text{ for every} \\ (n_1, \dots, n_k) \in \mathcal{F}_1 \text{ and scalars } \lambda_1, \dots, \lambda_k \end{array} \right\},$$

then  $\|\cdot\|_s^1$  is a norm on  $B_{1/4}(K)$  equivalent to the norm  $\|\cdot\|_{1/4}$ . This answers in the affirmative a question raised by Haydon, Odell and Rosenthal in [H-O-R]. From this result and (3) we have the characterization of functions in  $B_{1/4}(K) \setminus C(K)$  ( $K$  compact) as pointwise limits of sequences of continuous functions generating spreading models equivalent to the summing basis of  $c_0$ .

More generally, we define analogously the spaces  $S_\xi(K)$  and the norms  $\|\cdot\|_s^\xi$  on them, employing the higher order Schreier family  $\mathcal{F}_\xi$ , for  $1 \leq \xi < \omega_1$ , as defined by Alspach and Argyros ([A-A]). According to Proposition 3.4,  $(S_\xi(K), \|\cdot\|_s^\xi)$  are Banach spaces, which, for separable metric spaces  $K$ , constitute a decreasing hierarchy whose intersection is equal to  $D(K)$  (Theorem 3.8) and of course  $S_1(K) = B_{1/4}(K)$ . We further provide alternative descriptions of the spaces  $S_\xi(K)$ ,  $1 \leq \xi < \omega_1$ , and characterize the Baire-1 functions not belonging to  $S_\xi(K)$  (Theorem 3.11), employing the notion of a null-coefficient of order  $\xi$  sequence, defined in [F2].

Because of Mazur's theorem,  $S_\xi(K)$  is actually a Banach space invariant. That is, if  $X$  is a separable Banach space,  $x^{**} \in X^{**} \setminus X$ , and  $K = Ba(X^*, w^*)$ , then if

$f = x^{**}|K$ ,  $f \in S_\xi(K)$  if and only if there exists a sequence  $(x_n)$  in  $X$  such that  $(x_n)$  generates a spreading model of order  $\xi$  equivalent to  $(s_n)$  and converges in the  $w^*$ -topology to  $f$ . Moreover, then

$$|f|_s^\xi = \inf\{C > 0 : \text{there exists } (x_n) \subset X \text{ and such that } x_n \xrightarrow{w^*} f \\ \text{and } \|\sum_{i=1}^k \lambda_i x_{n_i}\|_\infty \leq C \|\sum_{i=1}^k \lambda_i s_i\| \text{ for every} \\ (n_i, \dots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \dots, \lambda_k\}.$$

A nice relation between the space  $(B_{1/4}(K), \|\cdot\|_{1/4})$  and the transfinite oscillations of a function is given in Theorem 2.9. Rosenthal in [R1] defined for every function  $f$  the  $\alpha^{\text{th}}$ -oscillation,  $\text{osc}_\alpha f$ , of  $f$  for every ordinal  $\alpha$  (cf. Definition 2.5). In [R2] the author proved the following structural result for  $D(K)$ : Let  $f$  be a real bounded function on an infinite metric space  $K$ . Then  $f \in D(K)$  if and only if there exist an ordinal  $\alpha$  such that  $\text{osc}_\alpha f$  is bounded and  $\text{osc}_\alpha f = \text{osc}_\beta f$  for all  $\beta > \alpha$ . Letting  $\tau$  be the least such  $\alpha$ , then

$$\|f\|_D = \| |f| + \text{osc}_\tau f \|_\infty \text{ for all } f \in D(K).$$

We prove an analogous structural result for the case of  $B_{1/4}(K)$ . Precisely, we have the following theorem: Let  $f$  be a real bounded function on a metric space  $K$ . Then  $f \in B_{1/4}(K)$  if and only if  $\text{osc}_\omega f$  is bounded. In this case

$$\|f\|_{1/4} = \| |f| + \widetilde{\text{osc}}_\omega f \|_\infty \text{ for } f \in B_{1/4}(K).$$

According to the principal result in [F2],  $\text{osc}_{\omega^\xi} f$  is bounded for every function  $f$  in  $S_\xi(K)$  and every ordinal  $\xi$ . It is an open problem whether the functions in  $S_\xi(K)$  are characterized by this property.

## 1. DIFFERENCES OF BOUNDED SEMICONTINUOUS FUNCTIONS

Let  $K$  be a metric space. We denote by  $C(K)$  the class of continuous functions on  $K$  and by  $B_1(K)$  the space of bounded first Baire class functions on  $K$  (i.e. the pointwise limits of uniformly bounded sequences of continuous functions).

An important subclass of  $B_1(K)$  is the class of differences of bounded semicontinuous functions on  $K$ , denoted by  $D(K)$ . It is easy to see that

$$D(K) = \left\{ f \in B_1(K) : f = u - v, \text{ where } u, v \geq 0 \text{ are bounded and} \right. \\ \left. \text{lower semicontinuous functions} \right\}.$$

The class  $D(K)$  is a Banach algebra with respect to the norm  $\|\cdot\|_D$ , defined as follows:

$$\|f\|_D = \inf \left\{ \|u + v\|_\infty : f = u - v \text{ for } u, v \geq 0, \text{ bounded and lower} \right. \\ \left. \text{semicontinuous functions} \right\}.$$

This infimum is attained according to [R1]. A result of Baire gives that

$$D(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f = \sum_n f_n \right. \\ \left. \text{pointwise and } \|\sum_n |f_n|\|_\infty < \infty \right\}$$

and it follows that

$$\|f\|_D = \inf \left\{ \left\| \sum_n |f_n| \right\|_\infty : (f_n) \subseteq C(K) \text{ and } f = \sum_n f_n \text{ pointwise} \right\}$$

for every  $f \in D(K)$  (see [R2]). It is easy to see that  $\|f\|_\infty \leq \|f\|_D$  for every  $f \in D(K)$  but the two norms are not equivalent in general.

In the following proposition we give the fundamental connection between the summing basis  $(s_n)$  of  $c_0$  and the functions in  $D(K)$ , as well as between  $(s_n)$  and the norm  $\|\cdot\|_D$ .

**1.1. Proposition.** *Let  $K$  be a metric space. Then*

$$D(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ and } C > 0 \text{ so that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty \leq C \left\| \sum_{i=1}^n \lambda_i s_i \right\| \text{ for all } n \in \mathbf{N} \text{ and scalars } \lambda_1, \dots, \lambda_n \right\},$$

where  $(s_n)$  is the summing basis of  $c_0$ . Also, for every  $f \in D(K)$ ,

$$\|f\|_D = \|f\|_s = \inf \left\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty \leq C \left\| \sum_{i=1}^n \lambda_i s_i \right\| \text{ for all } n \in \mathbf{N} \text{ and scalars } \lambda_1, \dots, \lambda_n \right\}.$$

*Proof.* If  $f \in D(K)$  then there exists a sequence  $(g_n)_{n=1}^\infty$  in  $C(K)$  such that  $f = \sum_{n=1}^\infty g_n$  and  $C = \left\| \sum_n |g_n| \right\|_\infty < \infty$ . Set  $f_n = \sum_{i=1}^n g_i$  for every  $n \in \mathbf{N}$ . Of course,  $f_n \rightarrow f$  pointwise and

$$\left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty = \left\| \sum_{i=1}^n (\lambda_i + \dots + \lambda_n) g_i \right\|_\infty \leq \left\| \sum_{i=1}^n |g_i| \right\|_\infty \cdot \left\| \sum_{i=1}^n \lambda_i s_i \right\| \leq C \cdot \left\| \sum_{i=1}^n \lambda_i s_i \right\|.$$

Hence,  $\|f\|_s \leq \|f\|_D$  for every  $f \in D(K)$ .

On the other hand, if there exist  $(f_n)$  in  $C(K)$  and  $C > 0$  such that  $f_n \rightarrow f$  pointwise and  $\left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty \leq C \left\| \sum_{i=1}^n \lambda_i s_i \right\|$  for every  $n \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_n$ , then if we set  $g_0 = 0$  and  $g_n = f_n - f_{n-1}$  for every  $n \in \mathbf{N}$ , we have that  $\sum_{n=1}^\infty g_n = f$ . Also, for  $x \in K$  and  $n \in \mathbf{N}$ ,

$$\begin{aligned} \sum_{i=1}^n |g_i|(x) &= \sum_{i=1}^n |f_i - f_{i-1}|(x) = \sum_{i=1}^n \varepsilon_i (f_i - f_{i-1})(x) \\ &= \left| \sum_{i=1}^n (\varepsilon_i - \varepsilon_{i+1}) f_i \right|(x) \leq C, \end{aligned}$$

where  $\varepsilon_i \in \{-1, 1\}$  so that  $\varepsilon_i (f_i - f_{i-1})(x) \geq 0$  for every  $i = 1, \dots, n$  and  $\varepsilon_{n+1} = 0$ . Hence, we have that  $\|f\|_D \leq \|f\|_s$  for every  $f \in D(K)$ .  $\square$

**1.2. Remark.** It is known ([H-O-R], [R1]) that, for a compact metric space  $K$ , every bounded sequence  $(f_n)$  in  $C(K)$  converging pointwise to a discontinuous function  $f$  has a basic subsequence  $(g_n)$  which dominates the summing basis  $(s_n)$  of  $c_0$ , i.e. there exists  $\mu > 0$  such that  $\mu \left\| \sum_{i=1}^n \lambda_i s_i \right\| \leq \left\| \sum_{i=1}^n \lambda_i g_i \right\|_\infty$  for every  $n \in \mathbf{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ . Hence, for a compact metric space  $K$ ,

$$D(K) \setminus C(K) = \left\{ f : K \rightarrow \mathbf{R} : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } (f_n) \text{ is equivalent to } (s_n) \right\}.$$

This result has been proved in [R1] also. Using Mazur's theorem we have that every uniformly bounded sequence  $(f_n)$  converging pointwise to a function  $f$  in  $D(K) \setminus C(K)$  has a convex block subsequence equivalent to  $(s_n)$ .

## 2. BAIRE-1/4 FUNCTIONS

As we mentioned before, the supremum norm is not equivalent, in general, to the  $\|\cdot\|_D$ -norm in  $D(K)$ . The closure of  $D(K)$  in  $(B_1(K), \|\cdot\|_\infty)$  has been denoted by  $B_{1/2}(K)$  in [H-O-R]. In the same paper the authors defined the subclass  $B_{1/4}(K)$  of  $B_1(K)$  as follows:

$$B_{1/4}(K) = \left\{ f \in B_1(K) : \text{there exists } (F_n) \subseteq D(K) \text{ such that } \|F_n - f\|_\infty \rightarrow 0 \text{ and } \sup_n \|F_n\|_D < \infty \right\}.$$

The space  $B_{1/4}(K)$  is complete with respect to the norm

$$\|f\|_{1/4} = \inf \left\{ \sup_n \|F_n\|_D : (F_n) \subseteq D(K) \text{ and } \|F_n - f\|_\infty \rightarrow 0 \right\}.$$

In the following theorem we will give a characterization of  $B_{1/4}(K)$  and we will define the  $\|\cdot\|_s^1$ -norm on it, in analogy to  $D(K)$  (Proposition 1.1). We will prove that this norm is equivalent to the  $\|\cdot\|_{1/4}$ -norm answering affirmatively the question raised by Haydon, Odell and Rosenthal in [H-O-R]. The techniques of this proof have been employed before in [F1]. The additional work here is to establish the relation between the norms. For completeness we give the proof in detail. We will use the Schreier family  $\mathcal{F}_1$  which is:

$$\mathcal{F}_1 = \left\{ (n_1, \dots, n_k) : k < n_1 < \dots < n_k \in \mathbf{N} \right\}.$$

**2.1. Theorem.** *Let  $K$  be a compact metric space. Then*

$$B_{1/4}(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ and } C > 0 \text{ so that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\| \text{ for every } (n_1, \dots, n_k) \in \mathcal{F}_1 \text{ and scalars } \lambda_1, \dots, \lambda_k \right\}.$$

Also, defining for  $f \in B_{1/4}(K)$

$$\|f\|_s^1 = \inf \left\{ C > 0 : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\| \text{ for every } (n_1, \dots, n_k) \in \mathcal{F}_1 \text{ and scalars } \lambda_1, \dots, \lambda_k \right\},$$

$\|\cdot\|_s^1$  is a norm on  $B_{1/4}(K)$  equivalent to the norm  $\|\cdot\|_{1/4}$ . Moreover,

$$\|f\|_s^1 \leq \|f\|_{1/4} \leq 4\|f\|_s^1 \text{ for every } f \in B_{1/4}(K).$$

*Proof.* Let  $f \in B_{1/4}(K)$ . According to the definition of  $(B_{1/4}(K), \|\cdot\|_{1/4})$ , for every  $\delta > 0$  there exists a sequence  $(F_m)$  in  $D(K)$  so that  $\|F_m - f\|_\infty \rightarrow 0$  and  $\sup_m \|F_m\|_D < \|f\|_{1/4} + \delta$ . Let  $M = \|f\|_{1/4} + \delta$  and  $(\epsilon_m)$  a decreasing sequence of positive numbers such that  $\epsilon_m < \frac{\delta}{2m}$  and  $\sum_{i=m+1}^\infty \epsilon_i < \epsilon_m$  for every  $m \in \mathbf{N}$ . We can assume that  $\|F_{m+1} - F_m\|_\infty < \epsilon_{m+1}$  for every  $m \in \mathbf{N}$ . Hence, for every  $m \in \mathbf{N}$  there exists a sequence  $(g_n^m)_{n=1}^\infty \subseteq C(K)$  converging pointwise to  $F_{m+1} - F_m$  and  $\|g_n^m\|_\infty < \epsilon_{m+1}$  for all  $n \in \mathbf{N}$ .

Since  $F_1 \in D(K)$ , by Proposition 1.1, there exists a sequence  $(f_n^1)$  in  $C(K)$  converging pointwise to  $F_1$  and satisfying

$$\left\| \sum_{i=1}^k \lambda_i f_i^1 \right\|_\infty \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for all  $k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$ . The sequence  $(f_n^1 + g_n^1)$  converges pointwise to  $F_2$ . Using Mazur's theorem and the fact that  $F_2 \in D(K)$ , we can find convex block subsequences  $(f_n^{1,2}), (g_n^{1,2})$  of  $(f_n^1), (g_n^1)$  respectively such that if  $f_n^2 = f_n^{1,2} + g_n^{1,2}$  for every  $n \in \mathbf{N}$  then  $f_n^2 \rightarrow F_2$  pointwise and

$$\left\| \sum_{i=1}^k \lambda_i f_i^2 \right\|_{\infty} \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every  $k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$ . Now, since  $f_n^2 + g_n^2$  converges to  $F_3$ , there exist convex block subsequences  $(f_n^{2,3}), (g_n^{2,3})$  of  $(f_n^2), (g_n^2)$  respectively, such that if  $f_n^3 = f_n^{2,3} + g_n^{2,3}$  for every  $n \in \mathbf{N}$  then  $f_n^3 \rightarrow F_3$  pointwise and

$$\left\| \sum_{i=1}^k \lambda_i f_i^3 \right\|_{\infty} \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every  $k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$ . Let  $(f_n^{1,2,3}), (g_n^{1,2,3})$  be the convex block subsequences of  $(f_n^{1,2}), (g_n^{1,2})$  respectively, such that  $f_n^{2,3} = f_n^{1,2,3} + g_n^{1,2,3}$  for every  $n \in \mathbf{N}$ . Hence  $f_n^3 = f_n^{1,2,3} + g_n^{1,2,3} + g_n^{2,3}$  for every  $n \in \mathbf{N}$ . We continue in the obvious way to construct  $f_n^{m,\dots,k}$  and  $g_n^{m,\dots,k}$  for every  $m, k, n \in \mathbf{N}$  with  $m \leq k$ , so that  $(g_n^{m,\dots,k}), (f_n^{m,\dots,k})$  to be convex block subsequences of  $(g_n^{m,\dots,l}), (f_n^{m,\dots,l})$  respectively for every  $m, l, k \in \mathbf{N}$  with  $m \leq l \leq k$  and

$$(*) \quad f_n^{m,\dots,k} = f_n^{m-1,m,\dots,k} + g_n^{m-1,m,\dots,k}$$

for every  $n, k, m \in \mathbf{N}$  with  $1 < m \leq k$ . Also, for every  $m \in \mathbf{N}$ , we construct the sequence  $(f_n^m)_{n=1}^{\infty}$  converging pointwise to  $F_m$  and

$$(**) \quad \left\| \sum_{i=1}^k \lambda_i f_i^m \right\|_{\infty} \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every  $k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$ . Finally, we set

$$h_n^m = f_n^{m,\dots,n} \text{ and } d_n^m = g_n^{m,\dots,n}$$

for every  $m, n \in \mathbf{N}$  with  $m \leq n$ .

Then, for every  $m \in \mathbf{N}$ ,  $(h_n^m)_{n=m}^{\infty}, (d_n^m)_{n=m}^{\infty}$  are convex block subsequences of  $(f_n^m)_{n=1}^{\infty}, (g_n^m)_{n=1}^{\infty}$  respectively, hence  $(h_n^m)_{n=m}^{\infty}$  converges pointwise to  $F_m$ ,  $\|d_n^m\|_{\infty} < \epsilon_{m+1}$  for every  $m, n \in \mathbf{N}$  with  $m \leq n$  and  $(d_n^m)_{n=m}^{\infty}$  converges pointwise to  $F_{m+1} - F_m$ . Also, according to (\*), we have that

$$h_n^m = h_n^{m-1} + d_n^{m-1} = h_n^l + d_n^l + \dots + d_n^{m-1}$$

for every  $n, m, l \in \mathbf{N}$  with  $l < m \leq n$ .

We set  $h_n = h_n^n$  for every  $n \in \mathbf{N}$ . Thus  $h_n = h_n^m + d_n^m + \dots + d_n^{n-1}$  for every  $m, n \in \mathbf{N}$  with  $m < n$ . It is easy to prove that  $(h_n)$  converges pointwise to  $f$ . If  $(n_1, \dots, n_k) \in \mathcal{F}_1$  and  $\lambda_1, \lambda_2, \dots, \lambda_k$  are scalars then

$$\left\| \sum_{i=1}^k \lambda_i h_{n_i} \right\|_{\infty} \leq \left\| \sum_{i=1}^k \lambda_i h_{n_i}^k \right\|_{\infty} + \left\| \sum_{i=1}^k \lambda_i (d_{n_i}^k + \dots + d_{n_i}^{n_i-1}) \right\|_{\infty}.$$

First, since  $(h_n^k)_{n=k}^{\infty}$  is a convex block subsequence of  $(f_n^k)_{n=1}^{\infty}$ , we have from (\*\*) that

$$\left\| \sum_{i=1}^k \lambda_i h_{n_i}^k \right\|_{\infty} \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|.$$

Secondly,

$$\begin{aligned} \left\| \sum_{i=1}^k \lambda_i (d_{n_i}^k + \cdots + d_{n_i}^{n_i-1}) \right\|_{\infty} &\leq \epsilon_k \cdot \sum_{i=1}^k |\lambda_i| \\ &\leq 2k\epsilon_k \left\| \sum_{i=1}^k \lambda_i s_i \right\| < \delta \left\| \sum_{i=1}^k \lambda_i s_i \right\|. \end{aligned}$$

Hence

$$\left\| \sum_{i=1}^k \lambda_i h_{n_i} \right\|_{\infty} \leq (\|f\|_{1/4} + 2\delta) \cdot \left\| \sum_{i=1}^k \lambda_i s_i \right\|.$$

This gives

$$\|f\|_s^1 \leq \|f\|_{1/4} + 2\delta \text{ for every } \delta > 0$$

and finally

$$\|f\|_s^1 \leq \|f\|_{1/4} \text{ for every } f \in B_{1/4}(K).$$

On the other hand, let  $(f_n)$  be a sequence in  $C(K)$  converging pointwise to  $f$  and  $C > 0$  such that

$$\left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_{\infty} \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every  $(n_1, \dots, n_k) \in \mathcal{F}_1$  and scalars  $\lambda_1, \dots, \lambda_k$ . According to a characterization of functions in  $B_{1/4}(K)$  given by Haydon, Odell and Rosenthal in [H-O-R], a function  $f$  belongs to  $B_{1/4}(K)$  if for  $\epsilon > 0$  there exists a sequence  $(g_n)_{n=0}^{\infty}$  in  $C(K)$  with  $g_0 = 0$ , converging pointwise to  $f$  and such that for every subsequence  $(g_{n_i})$  of  $(g_n)$  and  $x \in K$  to have

$$\sum_{j \in B((n_i), x)} |g_{n_{j+1}}(x) - g_{n_j}(x)| \leq M,$$

where

$$B((n_i), x) = \{j \in \mathbf{N} : |g_{n_{j+1}}(x) - g_{n_j}(x)| \geq \epsilon\}.$$

In this case, it is easy to see that  $\|f\|_{1/4} \leq 4M$ .

For  $\epsilon > 0$ , let  $m$  be an integer such that  $m > C/\epsilon$ . Set  $g_n = f_{2m+n}$  for every  $n \in \mathbf{N}$ . Then, for every strictly increasing sequence  $(n_i)$  in  $\mathbf{N}$  and  $x \in K$  we claim that  $\#B((n_i), x) < m$ . Indeed, if  $j_1, \dots, j_m \in B((n_i), x)$ , then

$$m \cdot \epsilon \leq \sum_{i=1}^m |g_{n_{j_i+1}}(x) - g_{n_{j_i}}(x)| = \sum_{i=1}^m \varepsilon_j (f_{2m+n_{j_i+1}} - f_{2m+n_{j_i}})(x) \leq C,$$

where  $\varepsilon_1, \dots, \varepsilon_m \in \{-1, 1\}$ , so that  $\varepsilon_j (f_{2m+n_{j_i+1}} - f_{2m+n_{j_i}})(x) \geq 0$ , a contradiction. Hence  $\#B((n_i), x) < m$  and thus

$$\sum_{j \in B((n_i), x)} |g_{n_{j+1}}(x) - g_{n_j}(x)| \leq C.$$

Hence  $f \in B_{1/4}(K)$  and  $\|f\|_{1/4} \leq 4\|f\|_s^1$ .  $\square$

**2.2. Remark.** It is easy to prove (see [F1]) that a sequence  $(x_n)$  in a Banach space  $X$  has a subsequence generating a spreading model equivalent to the summing basis  $(s_n)$  if and only if it has a subsequence  $(y_n)$  with the following property:

there exist  $\mu, C > 0$  such that

$$\mu \left\| \sum_{i=1}^k \lambda_i s_i \right\| \leq \left\| \sum_{i=1}^k \lambda_i y_{n_i} \right\| \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every  $(n_1, \dots, n_k) \in \mathcal{F}_1$  and scalars  $\lambda_1, \dots, \lambda_k$ .

Hence, it follows from the previous theorem and Remark 1.2, for a compact metric space  $K$  that

$$B_{1/4}(K) \setminus C(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \rightarrow f \right. \\ \left. \begin{array}{l} \text{pointwise and } (f_n) \text{ generates spreading} \\ \text{model equivalent to } (s_n) \end{array} \right\}.$$

This result has been proved in [F1] also. Furthermore, it has been proved in [H-O-R] that every uniformly bounded sequence  $(f_n)$  in  $C(K)$  converging pointwise to a function in  $B_{1/4}(K) \setminus C(K)$  has a convex block subsequence generating a spreading model equivalent to  $(s_n)$ .

In the following proposition we will give another description of  $B_{1/4}(K)$  and we will prove the equality of the norm  $\|\cdot\|_s^1$  with a norm on  $B_{1/4}(K)$  analogous to the  $\|\cdot\|_D$ -norm on  $D(K)$ .

**2.3. Proposition.** *For every compact metric space  $K$ , a function  $f: K \rightarrow \mathbf{R}$  belongs to  $B_{1/4}(K)$  if and only if there exists  $(f_n)$  in  $C(K)$  such that  $f = \sum_{n=1}^{\infty} f_n$  pointwise and for  $n_0 = f_0 = 0$ ,*

$$\sup \left\{ \left\| \sum_{i=1}^k |f_{n_{i-1}+1} + \dots + f_{n_i}| \right\|_{\infty} : (n_1, \dots, n_k) \in \mathcal{F}_1 \right\} < \infty.$$

Also, for every  $f \in B_{1/4}(K)$  we have

$$\|f\|_s^1 = \|f\|_D^1 = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^k |f_{n_{i-1}+1} + \dots + f_{n_i}| \right\|_{\infty} : (n_1, \dots, n_k) \in \mathcal{F}_1 \right\} : \right. \\ \left. (f_n) \subseteq C(K) \text{ with } f = \sum_{n=1}^{\infty} f_n \right\}.$$

*Proof.* If  $f \in B_{1/4}(K)$  then for every  $\epsilon > 0$ , from the previous theorem, there exists  $(g_n)_{n=0}^{\infty} \subseteq C(K)$ ,  $g_0 = 0$ , such that  $g_n \rightarrow f$  pointwise and

$$\left\| \sum_{i=1}^k \lambda_i g_{n_i} \right\|_{\infty} \leq (\|f\|_s^1 + \epsilon) \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every  $(n_1, \dots, n_k) \in \mathcal{F}_1$  and scalars  $\lambda_1, \dots, \lambda_k$ . Set  $f_n = g_n - g_{n-1}$  for every  $n \in \mathbf{N}$ . Then  $f = \sum_{n=1}^{\infty} f_n$  pointwise. Also, for  $(n_1, \dots, n_k) \in \mathcal{F}_1$  and  $x \in K$  we have

$$\begin{aligned} \sum_{i=1}^k |f_{n_{i-1}+1} + \dots + f_{n_i}|(x) &= \sum_{i=1}^k \varepsilon_i (f_{n_{i-1}+1} + \dots + f_{n_i})(x) \\ &= \left| \sum_{i=1}^k \varepsilon_i (g_{n_i} - g_{n_{i-1}}) \right|(x) = \left| \sum_{i=1}^k (\varepsilon_i - \varepsilon_{i+1}) g_{n_i} \right|(x) \leq \|f\|_s^1 + \epsilon, \end{aligned}$$



where  $\varepsilon_i \in \{-1, 1\}$  so that  $\varepsilon_i(f_{n_{i-1}+1} + \cdots + f_{n_i})(x) \geq 0$  for all  $i = 1, \dots, k$  and  $\varepsilon_{k+1} = 0$ . This gives that  $\|f\|_D^1 \leq \|f\|_s^1$  for every  $f \in B_{1/4}(K)$ .

On the other hand, let  $(g_n) \subseteq C(K)$  and  $C > 0$  be such that  $f = \sum_{n=1}^{\infty} g_n$  pointwise and

$$\left\| \sum_{i=1}^k |g_{n_{i-1}+1} + \cdots + g_{n_i}| \right\|_{\infty} \leq C \quad (n_0 = g_0 = 0)$$

for every  $(n_1, \dots, n_k) \in \mathcal{F}_1$ . Set  $f_n = \sum_{i=1}^n g_i$  for every  $n \in \mathbb{N}$ . Of course  $f_n \rightarrow f$  pointwise. Also, for  $(n_1, \dots, n_k) \in \mathcal{F}_1$ ,  $x \in K$  and scalars  $\lambda_1, \dots, \lambda_k$  we have

$$\begin{aligned} \left| \sum_{i=1}^k \lambda_i f_{n_i} \right| (x) &= \left| \sum_{i=1}^k \lambda_i (g_1 + \cdots + g_{n_i}) \right| (x) \\ &= \left| \sum_{i=1}^k (\lambda_i + \cdots + \lambda_k) \cdot (g_{n_{i-1}+1} + \cdots + g_{n_i}) \right| (x) \\ &\leq \sum_{i=1}^k \left| \sum_{j=i}^k \lambda_j \right| \cdot \left| \sum_{j=n_{i-1}+1}^{n_i} g_j \right| (x) \\ &\leq \left\| \sum_{i=1}^k \lambda_i s_i \right\| \cdot \left( \sum_{i=1}^k \left| \sum_{j=n_{i-1}+1}^{n_i} g_j \right| \right) (x) \leq C \cdot \left\| \sum_{i=1}^k \lambda_i s_i \right\|. \end{aligned}$$

Hence  $f \in B_{1/4}(K)$  and  $\|f\|_s^1 \leq \|f\|_D^1$ . This completes the proof.  $\square$

**2.4. Corollary.** *For every compact metric space  $K$ , a function  $f : K \rightarrow \mathbf{R}$  belongs to  $B_{1/4}(K)$  if and only if there exists  $(f_n)$  in  $C(K)$  such that  $f_n \rightarrow f$  pointwise and for  $n_0 = f_0 = 0$ ,*

$$\sup \left\{ \left\| \sum_{i=1}^k |f_{n_i} - f_{n_{i-1}}| \right\|_{\infty} : (n_1, \dots, n_k) \in \mathcal{F}_1 \right\} < \infty.$$

Also, for every  $f \in B_{1/4}(K)$  we have

$$\|f\|_s^1 = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^k |f_{n_i} - f_{n_{i-1}}| \right\|_{\infty} : (n_1, \dots, n_k) \in \mathcal{F}_1 \right\} : \right. \\ \left. \text{where } (f_n) \subseteq C(K) \text{ and } f_n \rightarrow f \text{ pointwise} \right\}.$$

In the following theorem we will give a characterization of the functions in  $B_{1/4}(K)$  and also an identity for  $\|f\|_{1/4}$ , where  $f$  is in  $B_{1/4}(K)$ , using the transfinite oscillations of  $f$ , which have been defined by H. Rosenthal in [R1]. We recall this definition.

**2.5. Definition.** [R1] Let  $K$  be a metric space. One defines the upper semicontinuous envelope  $\mathcal{U}g$  of an extended real valued function  $g : K \rightarrow [-\infty, +\infty]$  as follows:

$$\mathcal{U}g = \inf \{ h : K \rightarrow [-\infty, \infty] : h \text{ is continuous and } h \geq g \}.$$

It is easy to see that for  $x \in K$

$$\begin{aligned}\mathcal{U}g(x) &= \overline{\lim}_{y \rightarrow x} g(y) = \max \{L \in [-\infty, +\infty] : \exists x_n \rightarrow x, g(x_n) \rightarrow L\} \\ &= \inf \left\{ \sup_{y \in U} g(y) : U \text{ is a neighbourhood of } x \right\}.\end{aligned}$$

In [R1] the author associates with each bounded function  $f : K \rightarrow \mathbf{R}$  a transfinite increasing family  $(\text{osc}_\alpha f)_{1 \leq \alpha}$  of upper semicontinuous functions which are called  $\alpha^{\text{th}}$ -oscillations of  $f$ . They have been defined by induction as follows:

$$\text{osc}_0 f = 0.$$

If  $\text{osc}_\alpha f$  has been defined, then for every  $x \in K$

$$\widetilde{\text{osc}}_{\alpha+1} f(x) = \overline{\lim}_{y \rightarrow x} (|f(y) - f(x)| + \text{osc}_\alpha f(y))$$

and consequently

$$\text{osc}_{\alpha+1} f = \mathcal{U} \widetilde{\text{osc}}_{\alpha+1} f.$$

If  $\alpha$  is a limit ordinal and  $\text{osc}_\beta f$  has been defined for all  $\beta < \alpha$  then

$$\widetilde{\text{osc}}_\alpha f = \sup_{\beta < \alpha} \text{osc}_\beta f$$

and consequently

$$\text{osc}_\alpha f = \mathcal{U} \widetilde{\text{osc}}_\alpha f.$$

According to [R2], a bounded function  $f : K \rightarrow \mathbf{R}$  is in  $D(K)$  if and only if  $\text{osc}_\alpha f$  is a bounded function for every ordinal  $\alpha$ . In this case there exists an ordinal  $\alpha$  so that  $\text{osc}_\alpha f$  is bounded and  $\text{osc}_\alpha f = \text{osc}_\beta f$  for all  $\beta > \alpha$ . Moreover, letting  $\tau$  be the least such  $\alpha$ ,

$$\|f\|_D = \| |f| + \text{osc}_\tau f \|_\infty.$$

We will prove an analogous structural result for  $B_{1/4}(K)$ . Precisely, we will prove that a bounded function  $f$  is in  $B_{1/4}(K)$  if and only if  $\text{osc}_\omega f$  is bounded and when this occurs then

$$\|f\|_{1/4} = \| |f| + \widetilde{\text{osc}}_\omega f \|_\infty.$$

Before the proof of this theorem we will give three lemmas. In the first lemma we list some elementary relations which are used in the sequel.

**2.6. Lemma.** *Let  $f, g$  be bounded functions on a metric space  $K$  and  $\alpha$  an ordinal number.*

- (1) *If  $f \leq g$  then  $\mathcal{U}f \leq \mathcal{U}g$ .*
- (2)  *$\mathcal{U}(f + g) \leq \mathcal{U}f + \mathcal{U}g$ .*
- (3)  *$\mathcal{U}(f - \mathcal{U}g) = \mathcal{U}(\mathcal{U}f - \mathcal{U}g) \leq \mathcal{U}(f - g)$ .*
- (4)  *$\mathcal{U}f = f$  if and only if  $f$  is upper semicontinuous.*
- (5)  *$\text{osc}_\alpha f$  is an upper semicontinuous  $[0, +\infty]$ -valued function on  $K$ .*
- (6)  *$\text{osc}_\alpha t f = |t| \text{osc}_\alpha f$  for every  $t \in \mathbf{R}$ .*
- (7)  *$\text{osc}_\alpha(f + g) \leq \text{osc}_\alpha f + \text{osc}_\alpha g$ .*
- (8)  *$\text{osc}_\alpha(f + g) = \text{osc}_\alpha f$  if  $g$  is a continuous function on  $K$ .*
- (9) *If  $\text{osc}_\alpha f$  is bounded then  $\mathcal{U}(\text{osc}_\alpha f \pm f) \leq \widetilde{\text{osc}}_{\alpha+1} f \pm f$ .*

*Proof.* The assertions (1)-(8) are easily proved. We will prove (9).

Let  $x \in K$ . We may choose  $(y_n)$  a sequence in  $K$  tending to  $x$  such that

$$\mathcal{U}(\text{osc}_\alpha f + f)(x) = \lim_{n \rightarrow \infty} \text{osc}_\alpha f(y_n) + f(y_n).$$

Since the functions  $f$  and  $\text{osc}_\alpha f$  are bounded, we may assume without loss of generality that the limits

$$\lim_{n \rightarrow \infty} \text{osc}_\alpha f(y_n), \lim_{n \rightarrow \infty} |f(y_n) - f(x)|, \lim_{n \rightarrow \infty} f(y_n)$$

all exist. We then have that

$$\begin{aligned} \widetilde{\text{osc}}_{\alpha+1} f(x) &\geq \lim_{n \rightarrow \infty} (|f(y_n) - f(x)| + \text{osc}_\alpha f(y_n)) \\ &= \lim_{n \rightarrow \infty} |f(y_n) - f(x)| + \lim_{n \rightarrow \infty} \text{osc}_\alpha f(y_n) \\ &\geq \lim_{n \rightarrow \infty} (\text{osc}_\alpha f(y_n) + f(y_n)) - f(x) \\ &= \mathcal{U}(\text{osc}_\alpha f + f)(x) - f(x). \end{aligned}$$

Thus it is proved that  $\mathcal{U}(\text{osc}_\alpha f + f) \leq \widetilde{\text{osc}}_{\alpha+1} f + f$ . If instead of  $f$  we use  $-f$ , we have that  $\mathcal{U}(\text{osc}_\alpha f - f) \leq \widetilde{\text{osc}}_{\alpha+1} f - f$ , since  $\widetilde{\text{osc}}_\alpha f = \widetilde{\text{osc}}_\alpha(-f)$ .  $\square$

**2.7. Lemma.** *Let  $f : K \rightarrow \mathbf{R}$  be a bounded function. For every  $n \in \mathbf{N}$  we have that*

$$\mathcal{U}(\text{osc}_{n+2} f - \text{osc}_{n+1} f) \leq \mathcal{U}(\text{osc}_{n+1} f - \text{osc}_n f).$$

*Proof.* Using (3) of the previous lemma, we have that

$$\begin{aligned} \mathcal{U}(\text{osc}_{n+2} f - \text{osc}_{n+1} f) &= \mathcal{U}(\widetilde{\text{osc}}_{n+2} f - \text{osc}_{n+1} f) \\ &\leq \mathcal{U}(\widetilde{\text{osc}}_{n+2} f - \widetilde{\text{osc}}_{n+1} f), \quad \text{for every } n \in \mathbf{N}. \end{aligned}$$

Hence it is sufficient to prove that

$$\mathcal{U}(\widetilde{\text{osc}}_{n+2} f - \widetilde{\text{osc}}_{n+1} f) \leq \mathcal{U}(\text{osc}_{n+1} f - \text{osc}_n f) \quad \text{for every } n \in \mathbf{N}.$$

By (1) and (4) of the previous lemma, the proof of this lemma will be complete as soon as we prove that

$$\widetilde{\text{osc}}_{n+2} f - \widetilde{\text{osc}}_{n+1} f \leq \mathcal{U}(\text{osc}_{n+1} f - \text{osc}_n f) \quad \text{for every } n \in \mathbf{N}.$$

Case  $n = 0$ . We have for  $x \in K$ ,

$$\begin{aligned} \widetilde{\text{osc}}_2 f(x) - \widetilde{\text{osc}}_1 f(x) &= \overline{\lim}_{y \rightarrow x} (\text{osc}_1 f(y) + |f(y) - f(x)|) - \overline{\lim}_{y \rightarrow x} |f(y) - f(x)| \\ &\leq \overline{\lim}_{y \rightarrow x} \text{osc}_1 f(y) = \mathcal{U}(\text{osc}_1 f)(x) = \text{osc}_1 f(x) \end{aligned}$$

(since  $\text{osc}_1 f$  is upper semicontinuous).

In general for  $n > 0$ ,  $n \in \mathbf{N}$ , we have for  $x \in K$ ,

$$\begin{aligned} \widetilde{\text{osc}}_{n+2} f(x) - \widetilde{\text{osc}}_{n+1} f(x) &= \overline{\lim}_{y \rightarrow x} (\text{osc}_{n+1} f(y) + |f(y) - f(x)|) - \overline{\lim}_{y \rightarrow x} (\text{osc}_n f(y) + |f(y) - f(x)|) \\ &\leq \overline{\lim}_{y \rightarrow x} (\text{osc}_{n+1} f(y) - \text{osc}_n f(y)) = \mathcal{U}(\text{osc}_{n+1} f - \text{osc}_n f)(x). \end{aligned}$$

This completes the proof.

The following lemma was proved by A. Louveau ([F-L]). For completeness we give the proof.  $\square$

**2.8. Lemma.** [F-L] *Let  $(g_n)_{n=1}^\infty$  be a sequence of bounded, upper semicontinuous functions on a metric space  $K$  with  $g_0 = 0$ . If the sequence  $(\mathcal{U}(g_{n+1} - g_n))_{n=0}^\infty$  is decreasing, then  $\mathcal{U}(g_{n+1} - g_n) \leq \frac{1}{n+1} \cdot g_{n+1}$  for every  $n \in \mathbf{N}$ .*

*Proof.* For  $n = 0$ , it reduces to  $\mathcal{U}g_1 \leq g_1$ , which is trivial since  $g_1$  is upper semicontinuous. Suppose we know it for  $n$ . For the induction step, it suffices, since  $g_{n+2}$  is usc, to prove:

$$g_{n+2} - g_{n+1} \leq \frac{g_{n+2}}{n+2}; \quad \text{i.e., } g_{n+2} \leq \frac{g_{n+2}}{n+2} + g_{n+1}.$$

But since  $1 = \frac{1}{n+2} + \frac{n+1}{n+2}$ , it suffices to show

$$\frac{n+1}{n+2}g_{n+2} \leq g_{n+1}, \quad \text{i.e., } g_{n+2} \leq \frac{n+2}{n+1}g_{n+1} = g_{n+1} + \frac{1}{n+1}g_{n+1}.$$

But this follows immediately from the induction step.  $\square$

**2.9. Theorem.** *Let  $K$  be a metric space. Then*

$$B_{1/4}(K) = \left\{ f : K \rightarrow \mathbf{R} \text{ bounded} : \text{osc}_\omega f \text{ is bounded} \right\} \quad \text{and} \\ \|f\|_{1/4} = \left\| |f| + \widetilde{\text{osc}}_\omega f \right\|_\infty \quad \text{for all } f \in B_{1/4}(K).$$

*Proof.* Suppose  $f \in B_{1/4}(K)$ . It follows from the definition of  $B_{1/4}(K)$  that for every  $\epsilon > 0$  one has a sequence  $(g_n)$  in  $D(K)$  with  $\|g_n - f\|_\infty \rightarrow 0$  and  $\sup_n \|g_n\|_D < \|f\|_{1/4} + \epsilon$ . Set  $\epsilon_n = \|g_n - f\|_\infty$ . Then by induction on  $k$ ,

$$\text{osc}_k f \leq \text{osc}_k g_n + 2k\epsilon_n \quad \text{for every } k, n \in \mathbf{N}.$$

Hence

$$\begin{aligned} |f| + \text{osc}_k f &\leq |g_n| + \epsilon_n + \text{osc}_k g_n + 2k\epsilon_n \\ &\leq |g_n| + \text{osc}_\tau g_n + (2k+1)\epsilon_n \\ &\leq \|g_n\|_D + (2k+1)\epsilon_n \quad \text{for every } k, n \in \mathbf{N}. \end{aligned}$$

Letting first  $n \rightarrow \infty$  and then  $k \rightarrow +\infty$ , we get

$$|f| + \widetilde{\text{osc}}_\omega f \leq \sup_n \|g_n\|_D \leq \|f\|_{1/4} + \epsilon.$$

Since  $\epsilon$  is arbitrary, we have that

$$\| |f| + \widetilde{\text{osc}}_\omega f \|_\infty \leq \|f\|_{1/4}$$

and of course that  $\widetilde{\text{osc}}_\omega f$  and, consequently,  $\text{osc}_\omega f$  are bounded functions.

On the other hand, let  $f : K \rightarrow \mathbf{R}$  be a bounded function with  $\text{osc}_\omega f$  also bounded. Set

$$g_n = \frac{\lambda_n - \mathcal{U}(\text{osc}_n f - f)}{2} - \frac{\lambda_n - \mathcal{U}(\text{osc}_n f + f)}{2},$$

where  $\lambda_n = \| |f| + \text{osc}_n f \|_\infty$  for every  $n \in \mathbf{N}$ . Then  $g_n \in D(K)$  and

$$\begin{aligned} \|g_n\|_D &\leq \left\| \lambda_n - \frac{1}{2}\mathcal{U}(\text{osc}_n f - f) - \frac{1}{2}(\text{osc}_n f + f) \right\|_\infty \leq \\ &\leq \lambda_n \leq \| |f| + \widetilde{\text{osc}}_\omega f \|_\infty \quad \text{for every } n \in \mathbf{N}. \end{aligned}$$

The first inequality holds for every  $n \in \mathbf{N}$ , since from (1), (2) and (4) of Lemma 2.6 we have

$$\mathcal{U}(\text{osc}_n f - f) + \mathcal{U}(\text{osc}_n f + f) \geq 2\mathcal{U}(\text{osc}_n f) = 2\text{osc}_n f \geq 0 \quad \text{and}$$

$$\begin{aligned}
& \lambda_n - \frac{1}{2}\mathcal{U}(\text{osc}_n f - f) - \frac{1}{2}\mathcal{U}(\text{osc}_n f + f) \\
& \geq \lambda_n - \mathcal{U}(\text{osc}_n f + |f|) \geq \lambda_n - \|\mathcal{U}(\text{osc}_n f + |f|)\|_\infty \\
& = \lambda_n - \|\text{osc}_n f + |f|\|_\infty = 0.
\end{aligned}$$

If we could prove that  $\|g_n - f\|_\infty \rightarrow 0$ , then we would have that  $f \in B_{1/4}(K)$  and  $\|f\|_{1/4} \leq \| |f| + \widetilde{\text{osc}}_\omega f \|_\infty$ . Now, according to (9) of Lemma 2.6,

$$\begin{aligned}
g_n - f &= \frac{1}{2}\mathcal{U}(\text{osc}_n f + f) - \frac{1}{2}\mathcal{U}(\text{osc}_n f - f) - f \\
&\leq \frac{1}{2}(\widetilde{\text{osc}}_{n+1} f + f) - \frac{1}{2}(\text{osc}_n f - f) - f \\
&= \frac{1}{2}(\widetilde{\text{osc}}_{n+1} f - \text{osc}_n f) \leq \frac{1}{2}(\text{osc}_{n+1} f - \text{osc}_n f) \text{ for every } n \in \mathbf{N}.
\end{aligned}$$

On the other direction,

$$\begin{aligned}
g_n - f &= \frac{1}{2}\mathcal{U}(\text{osc}_n f + f) - \frac{1}{2}\mathcal{U}(\text{osc}_n f - f) - f \\
&\geq \frac{1}{2}(\text{osc}_n f + f) - \frac{1}{2}(\widetilde{\text{osc}}_{n+1} f - f) - f \\
&= -\frac{1}{2}(\widetilde{\text{osc}}_{n+1} f - \text{osc}_n f) \geq -\frac{1}{2}(\text{osc}_{n+1} f - \text{osc}_n f) \text{ for every } n \in \mathbf{N}.
\end{aligned}$$

Hence

$$|g_n - f| \leq \frac{1}{2}(\text{osc}_{n+1} f - \text{osc}_n f) \text{ for every } n \in \mathbf{N}.$$

According to Lemma 2.7, the sequence  $(\mathcal{U}(\text{osc}_{n+1} f - \text{osc}_n f))_{n=0}^\infty$  is decreasing. Hence, using Lemma 2.8, we have that

$$\begin{aligned}
|g_n - f| &\leq \frac{1}{2}(\text{osc}_{n+1} f - \text{osc}_n f) \leq \frac{1}{2}\mathcal{U}(\text{osc}_{n+1} f - \text{osc}_n f) \\
&\leq \frac{1}{n+1}\text{osc}_{n+1} f \leq \frac{1}{n+1}\text{osc}_\omega f \leq \frac{1}{n+1}\|\text{osc}_\omega f\|_\infty.
\end{aligned}$$

Thus  $\|g_n - f\|_\infty \leq \frac{1}{n+1} \cdot \|\text{osc}_\omega f\|_\infty$  and, finally,  $\|g_n - f\|_\infty \rightarrow 0$ . This finishes the proof of the theorem.  $\square$

**2.10. Remark.** Using the invariants  $(f_\alpha)_{1 \leq \alpha}$  which have been introduced by Kechris and Louveau in [K-L] and which are similar to the  $\alpha^{\text{th}}$ -oscillations of the function  $f$ , we proved with Louveau ([F-L]) that a bounded function  $f$  is in  $B_{1/4}(K)$  if and only if  $f_\omega$  is bounded and in this case

$$\frac{1}{3}\|f_\omega\|_\infty \leq \|f\|_{1/4} \leq 4\|f_\omega\|_\infty + 5\|f\|_\infty.$$

But the previous theorem shows that the transfinite oscillations appear to be more appropriate than the  $f_\alpha$ 's.

After proving this theorem, I learned that H. Rosenthal ([R2]) had an analogous result. Precisely, he proved in [R2] that  $f$  belongs to  $B_{1/4}(K)$  (case  $f : K \rightarrow \mathbf{C}$ ) if and only if  $\text{osc}_\omega f$  is bounded and when this occurs and  $f$  is real valued,

$$\frac{1}{2}(\|f\|_\infty + \|\text{osc}_\omega f\|_\infty) \leq \|f\|_{1/4} \leq \|f\|_\infty + 3\|\text{osc}_\omega f\|_\infty.$$

### 3. A CLASSIFICATION OF $B_{1/4}(K)$

We will define a classification of  $B_{1/4}(K)$ , where  $K$  is a separable metric space, into a decreasing hierarchy  $(S_\xi(K))_{1 \leq \xi < \omega_1}$  of Banach spaces whose intersection is equal to  $D(K)$ . The functions in  $S_\xi(K)$  have a property stronger than the one of the functions in  $B_{1/4}(K)$  which is described in Proposition 2.3. Precisely, the

families  $\mathcal{F}_\xi$ , which have been defined by D. Alspach and S. Argyros in [A-A], are used instead of the Schreier family  $\mathcal{F}_1$ . We quote the definition of the  $\mathcal{F}_\xi$ 's.

**3.1. Definition** ([A-A]). For every limit ordinal  $\xi$ , let  $(\xi_n)$  be a sequence of ordinal numbers strictly increasing to  $\xi$ . Then  $\mathcal{F}_0 = \{\{n\} : n \in \mathbf{N}\}$ .

Suppose that  $\mathcal{F}_\xi$  is defined, then

$$\mathcal{F}_{\xi+1} = \{F \subseteq \mathbf{N} : F \subseteq F_1 \cup \dots \cup F_n \text{ with } \{n\} < F_1 < \dots < F_n \text{ and } F_i \in \mathcal{F}_\xi \text{ for all } i = 1, \dots, n\}.$$

If  $\xi$  is a limit ordinal,  $\mathcal{F}_\xi = \{F \subseteq \mathbf{N} : F \in \mathcal{F}_{\xi_n} \text{ and } \{n\} \leq F\}$ .

Using the families  $\mathcal{F}_\xi$ , for every ordinal  $\xi$ , we extended the notion of spreading model in [F2] as follows:

**3.2. Definition** ([F2]). Let  $X$  be a Banach space,  $\xi$  an ordinal number and  $(x_n)$  a sequence in  $X$ . We say that  $(x_n)$  generates spreading model of order  $\xi$  equivalent to a basic sequence  $(e_n)$  if there exist  $\mu > 0$  and  $C > 0$  such that:

$$\mu \left\| \sum_{i=1}^k \lambda_i e_{n_i} \right\| \leq \left\| \sum_{i=1}^k \lambda_i x_{n_i} \right\| \leq C \left\| \sum_{i=1}^k \lambda_i e_{n_i} \right\|$$

for every  $(n_1, \dots, n_k) \in \mathcal{F}_\xi$  and scalars  $\lambda_1, \dots, \lambda_k$ .

Now we will define the spaces  $S_\xi(K)$  for every ordinal  $\xi$ , which are characterized by spreading models of order  $\xi$  equivalent to the summing basis  $(s_n)$  of  $c_0$ .

**3.3. Definition.** Let  $K$  be a metric space and  $\xi$  an ordinal number. We define the space

$$S_\xi(K) = \left\{ f : K \rightarrow \mathbf{R} : \text{there exists } (f_n) \subseteq C(K) \text{ and } C > 0 \text{ such that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\| \text{ for every } (n_1, \dots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \dots, \lambda_k \right\}$$

and the norm  $\|\cdot\|_s^\xi$  on it as follows:

$$\|f\|_s^\xi = \inf \left\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\| \text{ for every } (n_1, \dots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \dots, \lambda_k \right\}$$

If  $K$  is a compact metric space, it is easy to prove (see Remark 1.2) that

$$S_\xi(K) \setminus C(K) = \left\{ f : K \rightarrow \mathbf{R} : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } (f_n) \text{ generates spreading model of order } \xi \text{ equivalent to } (s_n) \right\}.$$

Of course,  $S_1(K) = B_{1/4}(K)$  for a compact metric space  $K$ . Also, for every ordinal number  $\xi$ ,  $S_\xi(K)$  is a linear subspace of  $B_1(K)$ . Although the family  $(\mathcal{F}_\xi)_{1 \leq \xi}$  is not increasing, it has the property: for every  $1 \leq \beta < \xi$ , there exists  $n_0 = n_0(\beta, \xi)$  in  $\mathbf{N}$  such that if  $A \in \mathcal{F}_\beta$  and  $\{n_0\} \leq A$  then  $A \in \mathcal{F}_\xi$ . Hence, it is easy to prove that the family  $(S_\xi(K))_{1 \leq \xi}$  is decreasing and, also,  $\|f\|_s^\beta \leq \|f\|_s^\xi$  for every  $1 \leq \beta < \xi$  and  $f$  in  $S_\xi(K)$ .

**3.4. Proposition.** For every ordinal number  $\xi$ ,  $(S_\xi(K), \|\cdot\|_s^\xi)$  is a Banach space.

*Proof.* Let  $\xi$  be an ordinal number and  $(F_n)$  a Cauchy sequence in  $(S_\xi(K), \|\cdot\|_s^\xi)$ . We can assume that  $\|F_{n+1} - F_n\|_s^\xi < \frac{1}{2^n}$  for every  $n \in \mathbf{N}$ . So, for every  $n \in \mathbf{N}$  we can find a sequence  $(\phi_m^n)_{m=1}^\infty \subseteq C(K)$  converging pointwise to  $F_{n+1} - F_n$  and satisfying

$$\left\| \sum_{i=1}^k \lambda_i \phi_{m_i}^n \right\|_\infty \leq \frac{1}{2^n} \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty$$

for every  $(m_1, \dots, m_k) \in \mathcal{F}_\xi$  and scalars  $\lambda_1, \dots, \lambda_k$ . Since  $\|f\|_\infty \leq \|f\|_s^\xi$  for every  $f \in S_\xi(K)$ , there exists  $F \in B_1(K)$  such that  $\|F_n - F\|_\infty \rightarrow 0$ .

Let  $n_0 \in \mathbf{N}$ . Set  $\Phi_n = F_{n+1} - F_n$  for every  $n \in \mathbf{N}$ , and  $f_n = \phi_{n_0}^{n_0} + \dots + \phi_n^n$  for every  $n \geq n_0$ . Then  $F - F_{n_0} = \sum_{n=n_0}^\infty \Phi_n$ . Also,  $f_n \rightarrow F - F_{n_0}$  pointwise. Indeed,

$$\|f_n - (\phi_{n_0}^{n_0} + \dots + \phi_n^n)\|_\infty = \|\phi_n^{l+1} + \dots + \phi_n^n\|_\infty \leq \sum_{i=l+1}^n \frac{1}{2^i} = \frac{1}{2^l}$$

for every  $n_0 \leq l < n \in \mathbf{N}$ . Hence, letting  $n \rightarrow \infty$ , we have for every  $x \in K$  and  $l \geq n_0$ ,

$$\Phi_{n_0}(x) + \dots + \Phi_l(x) - \frac{1}{2^l} \leq \liminf_n f_n(x) \leq \overline{\lim}_n f_n(x) \leq \Phi_{n_0}(x) + \dots + \Phi_l(x) + \frac{1}{2^l}.$$

Letting  $l \rightarrow \infty$ , this gives that  $f_n \rightarrow F - F_{n_0}$  pointwise.

On the other hand, for every  $(n_1, \dots, n_k) \in \mathcal{F}_\xi$  and scalars  $\lambda_1, \dots, \lambda_k$  we have that

$$\begin{aligned} \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_\infty &= \left\| \sum_{i=1}^k (\lambda_i \phi_{n_i}^{n_0} + \dots + \lambda_i \phi_{n_i}^{n_i}) \right\|_\infty \\ &\leq \sum_{j=n_0}^{n_1} \left\| \sum_{i=1}^k \lambda_i \phi_{n_i}^j \right\|_\infty + \sum_{j=n_1+1}^{n_2} \left\| \sum_{i=2}^k \lambda_i \phi_{n_i}^j \right\|_\infty + \dots + \sum_{j=n_{k-1}+1}^{n_k} |\lambda_k| \|\phi_{n_k}^j\|_\infty \\ &\leq \sum_{j=n_0}^{n_1} \frac{1}{2^j} \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty + \sum_{j=n_1+1}^{n_2} \frac{1}{2^j} \left\| \sum_{i=2}^k \lambda_i s_i \right\|_\infty + \dots + \sum_{j=n_{k-1}+1}^{n_k} \frac{1}{2^j} \|\lambda_k s_k\|_\infty \\ &\leq \left( \sum_{j=n_0}^\infty \frac{1}{2^j} \right) \cdot 2 \cdot \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty = \frac{1}{2^{n_0}} \cdot \left\| \sum_{i=1}^k \lambda_i s_i \right\|_\infty. \end{aligned}$$

Hence  $F - F_{n_0} \in S_\xi(K)$ , whence  $F \in S_\xi(K)$ . Also, we have that

$$\|F - F_{n_0}\|_s^\xi \leq \frac{1}{2^{n_0}} \text{ for every } n_0 \in \mathbf{N},$$

which gives that  $(F_n)$  converges to  $F$  with respect to the  $\|\cdot\|_s^\xi$ -norm. This completes the proof.  $\square$

We will give more descriptions of the spaces  $S_\xi(K)$  in analogy to  $B_{1/4}(K)$  (see Proposition 2.3 and Corollary 2.4).

**3.5. Proposition.** *For every metric space  $K$  and ordinal number  $\xi$ , a function  $f : K \rightarrow \mathbf{R}$  belongs to  $S_\xi(K)$  if and only if there exists  $(f_n)$  in  $C(K)$  such that  $f = \sum_{n=1}^\infty f_n$  pointwise and for  $n_0 = f_0 = 0$ ,*

$$\sup \left\{ \left\| \sum_{i=1}^k |f_{n_{i-1}+1} + \dots + f_{n_i}| \right\|_\infty : (n_1, \dots, n_k) \in \mathcal{F}_\xi \right\} < \infty.$$

Also, for every  $f \in S_\xi(K)$ ,

$$\|f\|_s^\xi = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^k |f_{n_{i-1}+1} + \cdots + f_{n_i}| \right\|_\infty : (n_1, \dots, n_k) \in \mathcal{F}_\xi \right\} : \right. \\ \left. \text{for every } (f_n) \text{ in } C(K) \text{ with } f = \sum_n f_n \text{ pointwise} \right\}.$$

*Proof.* The proof is analogous to the proof of Proposition 2.3.  $\square$

**3.6. Corollary.** For every metric space  $K$  and ordinal number  $\xi$ , a function  $f : K \rightarrow \mathbf{R}$  belongs to  $S_\xi(K)$  if and only if there exists  $(f_n)$  in  $C(K)$  such that  $f_n \rightarrow f$  pointwise and for  $n_0 = f_0 = 0$ ,

$$\sup \left\{ \left\| \sum_{i=1}^k |f_{n_i} - f_{n_{i-1}}| \right\|_\infty : (n_1, \dots, n_k) \in \mathcal{F}_\xi \right\} < \infty.$$

Also, for every  $f \in S_\xi(K)$ ,

$$\|f\|_s^\xi = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^k |f_{n_i} - f_{n_{i-1}}| \right\|_\infty : (n_1, \dots, n_k) \in \mathcal{F}_\xi \right\} : \right. \\ \left. \text{for every } (f_n) \subseteq C(K) \text{ with } f_n \rightarrow f \text{ pointwise} \right\}.$$

From a result in [F2], we have the following connection between the functions in  $S_\xi(K)$  and the transfinite oscillations.

**3.7. Theorem** ([F2]). Let  $K$  be a metric space and  $\xi$  an ordinal number. Then

$$S_\xi(K) \subseteq \left\{ f : K \rightarrow \mathbf{R} : \text{osc}_{\omega^\xi} f \text{ is bounded} \right\}.$$

*Proof.* It follows from the proof of Theorem 9 in [F2] that, for every function  $f$  in  $S_\xi(K)$ , the function  $u_{\omega^\xi}(f)$  is bounded (the functions  $u_\alpha(f)$ , were introduced in [R1] and are similar to the  $\alpha^{\text{th}}$ -oscillations of  $f$ ). But, as it is proved in [R1],

$$\text{osc}_\alpha f \leq u_\alpha(f) + u_\alpha(-f)$$

for every ordinal number  $\alpha$ . Hence,  $\text{osc}_{\omega^\xi} f$  is bounded.

This theorem yields immediately the following result.  $\square$

**3.8. Theorem.** Let  $K$  be a separable metric space. The intersection of all the classes  $S_\xi(K)$ ,  $1 \leq \xi < \omega_1$ , is equal to  $D(K)$ .

*Proof.* It follows from the previous theorem and the fact that  $f$  belongs to  $D(K)$  if and only if  $\text{osc}_\alpha f$  is bounded for every countable ordinal  $\alpha$  ([R1]).  $\square$

In [F2] we defined for every ordinal  $\xi$  the notion of a null-coefficient of order  $\xi$  ( $\xi$ -n.c.) sequence in a Banach space and we proved that every bounded, Baire-1 function  $f$  with  $\text{osc}_{\omega^\xi} f$  unbounded has the property that every bounded sequence of continuous functions converging pointwise to  $f$  is null-coefficient of order  $\xi$ . We will prove in the sequel that this property characterizes the functions in  $B_1(K) \setminus S_\xi(K)$ .

**3.9. Definition** ([F2]). A sequence  $(x_n)$  in a Banach space is called null-coefficient of order  $\xi$  ( $\xi$ -n.c.), where  $\xi$  is an ordinal number, if whenever the scalars  $(c_n)$  satisfy:

$$\sup \left\{ \left\| \sum_{i=1}^k c_{n_{2i}} (x_{n_{2i}} - x_{n_{2i-1}}) \right\| : (n_1, \dots, n_{2k}) \in \mathcal{F}_\xi \right\} < \infty$$

the sequence  $(c_n)$  converges to 0.



**3.10. Proposition.** *Let  $\xi$  be an ordinal number, and  $(x_n)$  a weak-Cauchy and non-weakly convergent sequence in a Banach space. Then  $(x_n)$  is not null-coefficient of order  $\xi$  if and only if it has a subsequence with spreading model of order  $\xi$  equivalent to the summing basis of  $c_0$ .*

*Proof.* If  $(x_n)$  is not null-coefficient of order  $\xi$  then there exists a bounded sequence of scalars  $(c_n)$  such that  $(c_n)$  is not null-converging and

$$(*) \quad \left\| \sum_{i=1}^k c_{n_{2i}} (x_{n_{2i}} - x_{n_{2i-1}}) \right\| \leq 1$$

for every  $(n_1, \dots, n_{2k}) \in \mathcal{F}_\xi$ .

So we can find  $\epsilon > 0$  and a subsequence  $(c_{n_t})$  of  $(c_n)$  such that  $c_{n_t} > \epsilon$  for every  $t \in \mathbf{N}$  (otherwise replace  $c_n$  by  $-c_n$ ).

Consider  $x_n$ ,  $n \in \mathbf{N}$ , as elements of  $C(K)$ , where  $K$  is the unit ball of the dual of  $X = [x_n]$ , the closed subspace generated by  $(x_n)$ , with respect to the weak\*-topology. Since  $(x_n)$  converges with respect to the  $w^*$ -topology to a function  $x^{**} \in X^{**} \setminus X$  (Remark 1.2) there exists a subsequence  $(x_{n_{t_s}})$  of  $(x_{n_t})$  and  $\mu > 0$  such that

$$\mu \left\| \sum_{i=1}^k \lambda_i s_i \right\| \leq \left\| \sum_{i=1}^k \lambda_i x_{n_{t_i}} \right\|$$

for every  $k \in \mathbf{N}$  and scalars  $\lambda_1, \dots, \lambda_k$ . Set  $y_s = x_{n_{t_s}}$  and  $c_{n_{t_s}} = a_s$  for every  $s \in \mathbf{N}$ .

We will prove that the subsequence  $(y_s)$  of  $(x_n)$  has spreading model of order  $\xi$  equivalent to the summing basis  $(s_n)$  of  $c_0$ . Indeed, for every  $(s_1, \dots, s_k) \in \mathcal{F}_\xi$  and  $x \in K$  we have  $y_{s_0} = y_0 = 0$  and

$$\begin{aligned} \sum_{i=1}^k |y_{s_i} - y_{s_{i-1}}|(x) &\leq \frac{1}{\epsilon} \sum_{i=1}^k a_{s_i} |y_{s_i} - y_{s_{i-1}}|(x) \\ &= \left| \frac{1}{\epsilon} \sum_{i=1}^k a_{s_i} \cdot \varepsilon_{s_i} (y_{s_i} - y_{s_{i-1}}) \right|(x) \quad (\text{where } \varepsilon_{s_i} \in \{-1, 1\}) \\ &\leq \frac{1}{\epsilon} a_{s_1} \|y_{s_1}\| + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ odd} \\ \varepsilon_{s_i}=1}}^k a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right|(x) \\ &\quad + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ odd} \\ \varepsilon_{s_i}=-1}}^k a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right|(x) + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ even} \\ \varepsilon_{s_i}=1}}^{\infty} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right|(x) \\ &\quad + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ even} \\ \varepsilon_{s_i}=-1}}^{\infty} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right|(x) \leq \frac{4}{\epsilon} + \frac{1}{\epsilon} \cdot \|(c_n)\|_\infty \cdot \|(\|x_n\|)\|_\infty = C. \end{aligned}$$

In the last inequality we used  $(*)$  and the fact that every subset  $H$  of a set  $F$  belonging to  $\mathcal{F}_\xi$  is in  $\mathcal{F}_\xi$  as well and that  $(n_{t_{s_1}}, \dots, n_{t_{s_k}}) \in \mathcal{F}_\xi$  for every  $(s_1, \dots, s_k)$  in  $\mathcal{F}_\xi$ .

Finally, for every  $(s_1, \dots, s_k) \in \mathcal{F}_\xi$  and scalars  $\lambda_1, \dots, \lambda_k$  we have

$$\left\| \sum_{i=1}^k \lambda_i y_{s_i} \right\| = \left\| \sum_{i=1}^k (\lambda_i + \dots + \lambda_k)(y_{s_i} - y_{s_{i-1}}) \right\| \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|,$$

which completes the proof.  $\square$

**3.11. Theorem.** *Let  $K$  be a metric space and  $\xi$  an ordinal number. Then*

$$B_1(K) \setminus S_\xi(K) = \left\{ f \in B_1(K) : \text{every bounded sequence } (f_n) \text{ in } C(K) \text{ converging pointwise to } f \text{ is null-coefficient of order } \xi \right\}.$$

*Proof.* Let  $f \in B_1(K) \setminus S_\xi(K)$  and a bounded sequence  $(f_n)$  in  $C(K)$  converging pointwise to  $f$ . Then  $(f_n)$  is null-coefficient of order  $\xi$ . Indeed, if  $(f_n)$  is not  $\xi$ -n.c., then according to the proof of the previous proposition, we can find a subsequence  $(g_n)$  of  $(f_n)$  and  $C > 0$  such that

$$\left\| \sum_{i=1}^k |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C$$

for all  $(n_1, \dots, n_k) \in \mathcal{F}_\xi$ . Hence, it follows from Corollary 3.6 that  $f \in S_\xi(K)$ , a contradiction.

On the other hand, if  $f \in S_\xi(K)$  then there exists a sequence  $(f_n) \subseteq C(K)$  converging pointwise to  $f$  and  $C > 0$  such that

$$\left\| \sum_{i=1}^k |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C$$

for every  $(n_1, \dots, n_k) \in \mathcal{F}_\xi$ , according to Corollary 3.6. Thus, if  $c_n = 1$  for every  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \left\| \sum_{i=1}^k (f_{n_{2i}} - f_{n_{2i-1}}) \right\|_\infty &\leq \left\| \sum_{i=1}^k |f_{n_{2i}} - f_{n_{2i-1}}| \right\|_\infty \\ &\leq \left\| \sum_{i=1}^{2k} |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C \end{aligned}$$

for every  $(n_1, \dots, n_{2k}) \in \mathcal{F}_\xi$ . Hence  $(f_n)$  is not null-coefficient of order  $\xi$ .

This completes the proof.  $\square$

**3.12. Corollary.** *Let  $K$  be a compact metric space. Then*

$$B_1(K) \setminus B_{1/4}(K) = \left\{ f \in B_1(K) : \text{every bounded sequence } (f_n) \text{ in } C(K) \text{ converging pointwise to } f \text{ is null-coefficient of order } 1 \right\}.$$

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